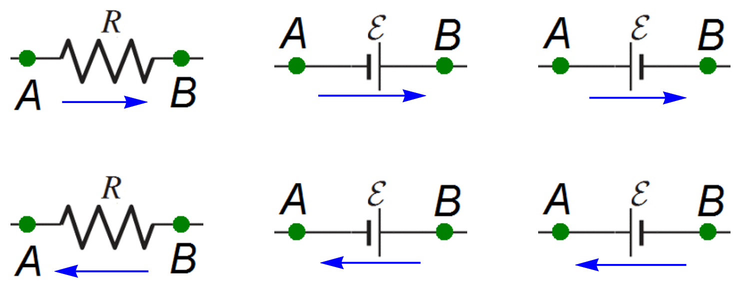
Lecture 16 - Circuit Problems

A Puzzle...

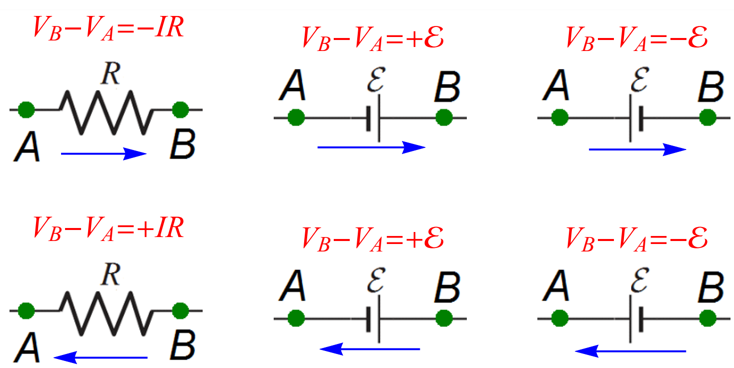
Crash Course in Circuits

Compute the change in voltage from point A to point B (in other words, the voltage difference $V_B - V_A$) in the following cases. Current of magnitude I flows in the direction given by the blue arrow.



<u>Solution</u>

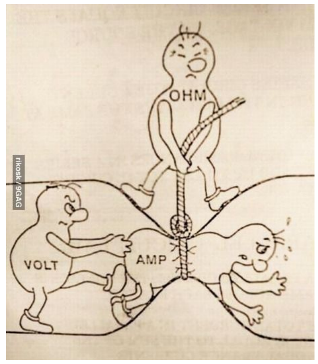
The voltage drop across a resistor is $I\,R$ in the direction of the current. The voltage drop across a battery is a fixed value (independent of current) which only depends on where the positive and negative terminals are (the positive terminal is represented by the longer line). Therefore, the corresponding voltage differences equal.



Any circuit consisting of only resistors and batteries can be analyzed using the above rules. □

Circuit Problems

Out[ ]=

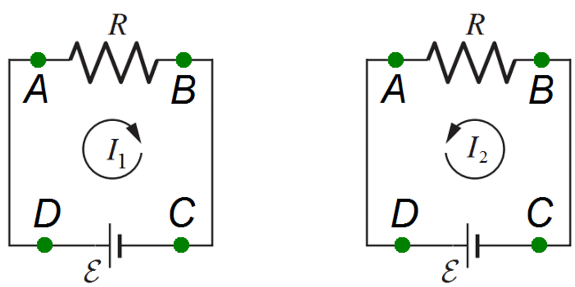


Basics

This is your one and only simple circuit problem. Enjoy it while it lasts!

Example

The two circuits below are identical, but we guess that the current travels clockwise on the left-hand side and counter-clockwise on the right-hand side. By considering the voltage drop going *clockwise* around the circuit, analyze both circuits and determine the magnitude and direction of the current. Then repeat the analysis going *counter-clockwise* around the circuit.



Solution

On the left-hand circuit, we move clockwise around the circuit. The voltage drop from A to B equals $-I_1 R$, and the voltage drop from C to D equals $\mathcal{E}$. Therefore, the loop equation equals

$$-I_1 R + \mathcal{E} = 0 \quad (1)$$

so that the current is

$$I_1 = \frac{\mathcal{E}}{R} \quad (2)$$

in the direction shown.

On the right-hand circuit, moving clockwise yields the loop equation

$$I_2 R + \mathcal{E} = 0 \quad (3)$$

so that

$$I_2 = -\frac{\mathcal{E}}{R} \quad (4)$$

indicating that we chose the wrong direction for the current (and therefore in agreement with Equation (2)).

Repeating the analysis on the left-hand circuit but moving counter-clockwise around the circuit, the loop equation equals

$$I_1 R - \mathcal{E} = 0 \tag{5}$$

which is the same result as Equation (1) and therefore also yields $I_1 = \frac{\mathcal{E}}{R}$.

Lastly, the right-hand circuit moving counter-clockwise yields

$$-I_2 R - \mathcal{E} = 0 \tag{6}$$

which is the same as Equation (3) and yields $I_2 = -\frac{\mathcal{E}}{R}$.

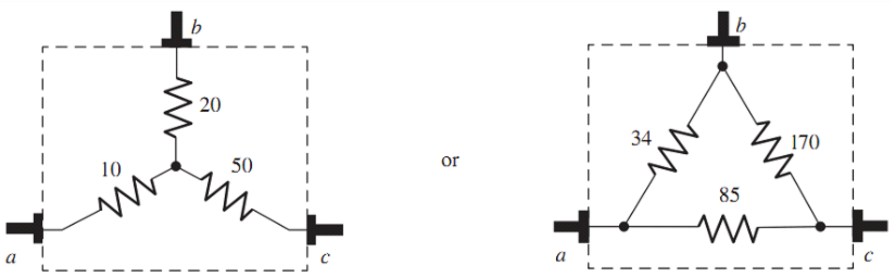
As expected, no matter which direction you travel around the circuit and which direction you predict that the current flows, you will find that the current has magnitude $\frac{\mathcal{E}}{R}$ and travels clockwise around the circuit.

Equivalent Boxes

Example

A black box with three terminals, a, b, and c, contains nothing but resistors and connecting wire. Measuring the resistance between pairs of terminals, we find $R_{ab} = 30\ \Omega$, $R_{ac} = 60\ \Omega$, and $R_{bc} = 70\ \Omega$. Show that the contents of the box could be either of the configurations shown in the figure below. Is there any other possibility? Are the two boxes completely equivalent, or is there an external measurement that would distinguish between them?

Out[ ]=



Solution

In the first case we simply have resistors in series so that

$$R_{ab} = 10\ \Omega + 20\ \Omega = 30\ \Omega \tag{7}$$
$$R_{ac} = 10\ \Omega + 50\ \Omega = 60\ \Omega \tag{8}$$
$$R_{bc} = 20\ \Omega + 50\ \Omega = 70\ \Omega \tag{9}$$

as desired. In the second case, the resistance between a and b is the sum of the parallel resistors $34\ \Omega$ and $85\ \Omega + 170\ \Omega = 255\ \Omega$, which is $\frac{1}{R_{ab}} = \frac{1}{34\ \Omega} + \frac{1}{255\ \Omega} = \frac{1}{30\ \Omega}$ or $R_{ab} = 30\ \Omega$. Repeating this calculation between all the terminals,

$$\frac{1}{R_{ab}} = \frac{1}{34\ \Omega} + \frac{1}{85\ \Omega + 170\ \Omega} = \frac{1}{30\ \Omega} \tag{10}$$
$$\frac{1}{R_{ac}} = \frac{1}{85\ \Omega} + \frac{1}{34\ \Omega + 170\ \Omega} = \frac{1}{60\ \Omega} \tag{11}$$
$$\frac{1}{R_{bc}} = \frac{1}{170\ \Omega} + \frac{1}{85\ \Omega + 34\ \Omega} = \frac{1}{70\ \Omega} \tag{12}$$

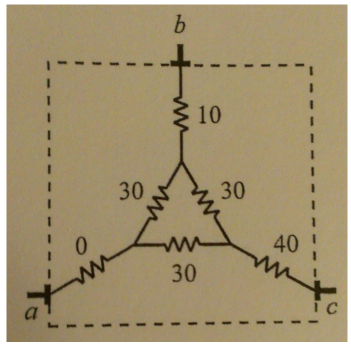
This proves that both of the boxes yield the desired resistances between the terminals.

For the two configurations shown, the above resistances are the only possibility that leads to the given resistances between the terminals. Each configuration yields three equations for the three resistances, so they are uniquely determined. You can prove this to yourself by solving the general system of equations using *Mathematica* and noting that there is only one solution.

```
Simplify@Solve[{(R1*(R2+R3))/(R1+R2+R3) == Rab, (R2*(R1+R3))/(R1+R2+R3) == Rac, (R3*(R2+R1))/(R1+R2+R3) == Rbc}, {R1, R2, R3}]
```

$$\left\{\left\{R1 \to \frac{Rab^2 + (Rac - Rbc)^2 - 2\,Rab\,(Rac + Rbc)}{2\,(Rab - Rac - Rbc)},\ R2 \to -\frac{Rab^2 + (Rac - Rbc)^2 - 2\,Rab\,(Rac + Rbc)}{2\,(Rab - Rac + Rbc)},\ R3 \to -\frac{Rab^2 + (Rac - Rbc)^2 - 2\,Rab\,(Rac + Rbc)}{2\,(Rab + Rac - Rbc)}\right\}\right\}$$

However, there are other configurations that also work, for example, the one pictured below (as you can check).



The potential assumed by the free terminal when the potentials at the other two terminals are fixed is the same for the two boxes. For example, if the potentials at b and c in the first box are fixed at ϕ_b and ϕ_c, then

$$\phi_b = \phi_c - I\,(70\ \Omega) \tag{13}$$

$$\phi_a = \phi_c - I\,(50\ \Omega) \tag{14}$$

Note that no current flows across the 10 Ω resistor in this case. Substituting the first equation into the second to get rid of I,

$$\phi_a = \tfrac{5}{7}\,\phi_b + \tfrac{2}{7}\,\phi_c \tag{15}$$

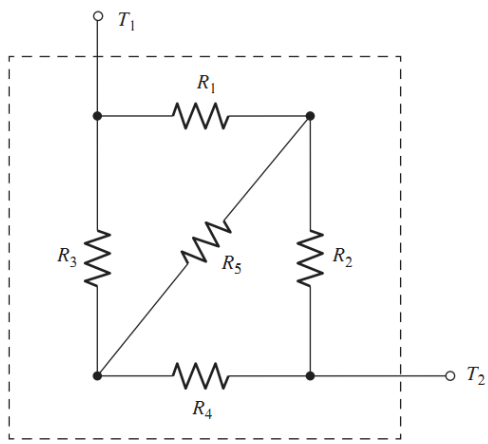
Therefore, the potential at a divides the difference $\phi_b - \phi_c$ in the ratio of 20 to 50. And in the second box the ratio is 34 to 85, which is the same. The two boxes are therefore indistinguishable by external measurements (using direct currents). You can show that the extra configuration shown above also yields the same ratio of 2 to 5. □

Using Symmetry

<u>Example</u>

This exercise deals with the equivalent resistance R_{eq} between terminals T_1 and T_2 for the network of five resistors shown below. One way to derive a formula for R_{eq} would be to solve the network for the current I that flows in at T_1 for a given voltage difference V between T_1 and T_2; then $R_{eq} = \frac{V}{I}$. The solution involves rather tedious algebra in which it is easy to make a mistake (although it is quick and painless if you use *Mathematica*), so we'll tell you most of the answer:

$$R_{eq} = (R_1 R_2 R_3 + R_1 R_2 R_4 + [?] + R_2 R_3 R_4 + R_5(R_1 R_3 + R_2 R_3 + [?] + R_2 R_4)) / (R_1 R_2 + R_1 R_4 + [?] + R_3 R_4 + R_5(R_1 + R_2 + R_3 + R_4)) \tag{16}$$



By considering the symmetry of the network you should be able to fill in the three missing terms. Now check the formula by directly calculating R_{eq} in four special cases:

(A) $R_5 = 0$

(B) $R_5 = \infty$

(C) $R_1 = R_3 = 0$

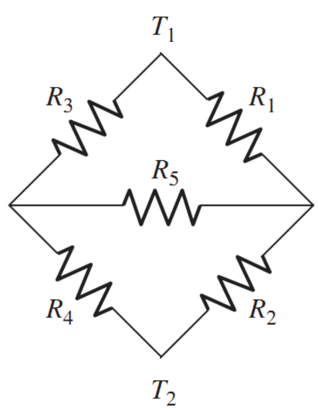
(D) $R_1 = R_2 = R_3 = R_4 = R$

Compare your results with what the formula gives.

Solution

First rule of problem solving: Don't get tricked by purposefully crappy diagrams; draw your own version!

The symmetry becomes much more apparent if we redraw the diagram above as

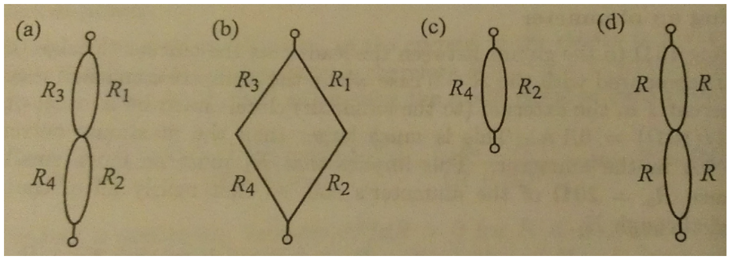


Looking at the solution

$$R_{eq} = (R_1 R_2 R_3 + R_1 R_2 R_4 + [?] + R_2 R_3 R_4 + R_5(R_1 R_3 + R_2 R_3 + [?] + R_2 R_4)) / (R_1 R_2 + R_1 R_4 + [?] + R_3 R_4 + R_5(R_1 + R_2 + R_3 + R_4)) \quad (17)$$

we see that the first missing term in the numerator must be $R_1 R_3 R_4$ (the mirror-image of the $R_1 R_2 R_3$ term) and the second term must be $R_1 R_4$ (the mirror-image of the $R_2 R_3$ term). The missing term in the denominator must be $R_2 R_3$ (the mirror-image of the $R_1 R_4$ term). Thus, the solution is

$$R_{eq} = (R_1 R_2 R_3 + R_1 R_2 R_4 + R_1 R_3 R_4 + R_2 R_3 R_4 + R_5(R_1 R_3 + R_2 R_3 + R_1 R_4 + R_2 R_4)) / (R_1 R_2 + R_1 R_4 + R_2 R_3 + R_3 R_4 + R_5(R_1 + R_2 + R_3 + R_4)) \tag{18}$$



(a) If $R_5 = 0$, we can redraw the diagram as shown above with the middle point collapsed (because it is short-circuited). The parallel combination of R_1 and R_3 is in series with the parallel combination of R_2 and R_4. Thus the resistance equals

$$R_{eq} = \frac{R_1 R_3}{R_1+R_3} + \frac{R_2 R_4}{R_2+R_4} = \frac{R_1 R_2 R_3+R_1 R_2 R_4+R_1 R_3 R_4+R_2 R_3 R_4}{R_1 R_2+R_1 R_4+R_2 R_3+R_3 R_4} \tag{19}$$

which agrees with Equation (18) when $R_5 = 0$.

(b) If $R_5 = \infty$, no current flows through the R_5 edge so we can redraw our circuit as shown above. The series combination of R_1 and R_2 is in parallel with the series combination of R_3 and R_4, so that the resistance equals

$$R_{eq} = \frac{(R_1+R_2)(R_3+R_4)}{R_1+R_2+R_3+R_4} \tag{20}$$

which agrees with Equation (18) when $R_5 = \infty$.

(c) If $R_1 = R_3 = 0$, then we effectively have the circuit shown above with R_5 short-circuited via R_1 and R_3. Therefore the resistance is simply R_2 and R_4 in parallel,

$$R_{eq} = \frac{R_2 R_4}{R_2+R_4} \tag{21}$$

in agreement with Equation (18) when all terms containing R_1 and R_3 are dropped.

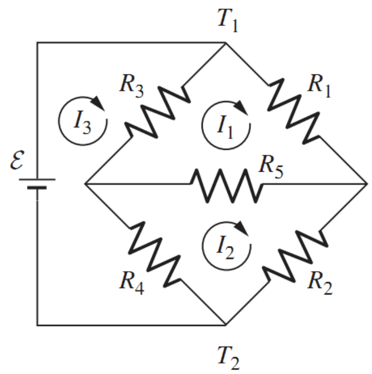
(d) If $R_1 = R_2 = R_3 = R_4 = R$, then by symmetry no current flows through R_5 (equivalently, the current will be split into $\frac{I}{2}$ going through R_1 and $\frac{I}{2}$ going through R_3, so that the voltage on either side of R_5 is the same, and therefore no current flows through R_5). Therefore we can redraw the circuit as shown above, with the effective resistance across R_1 and R_3 equal to $\frac{R}{2}$, and likewise for R_2 and R_4. Therefore, the effective resistance equals

$$R_{eq} = R \tag{22}$$

in agreement with Equation (18) in this limit.

More generally, if $R_1 = R_3 = a$ and $R_2 = R_4 = b$, then no current flows through R_5, so we again have two sets of parallel resistors. You should check that the formula agrees with what you calculate directly.

Finally, we can compute Equation (18) directly by using the loop equations for the circuit below.



The three loop equations are

$$\begin{aligned} \mathcal{E} - (I_3 - I_1)\, R_3 - (I_3 - I_2)\, R_4 &= 0 \\ -I_1\, R_1 - (I_1 - I_2)\, R_5 - (I_1 - I_3)\, R_3 &= 0 \\ -I_2\, R_2 - (I_2 - I_3)\, R_4 - (I_2 - I_1)\, R_5 &= 0 \end{aligned} \tag{23}$$

Solving these equations using *Mathematica*,

```
FullSimplify@
 Solve[{ℰ - (I3 - I1) R3 - (I3 - I2) R4 == 0, -I1 R1 - (I1 - I2) R5 - (I1 - I3) R3 == 0, -I2 R2 - (I2 - I3) R4 - (I2 - I1) R5 == 0}, {I1, I2, I3}]
```

```
{{I1 → ((R3 (R2 + R4) + (R3 + R4) R5) ℰ) / (R1 R2 R3 + R1 R2 R4 + R1 R3 R4 + R2 R3 R4 + (R1 + R2) (R3 + R4) R5),
  I2 → (((R1 + R3) R4 + (R3 + R4) R5) ℰ) / (R1 R2 R3 + R1 R2 R4 + R1 R3 R4 + R2 R3 R4 + (R1 + R2) (R3 + R4) R5),
  I3 → (((R1 + R3) (R2 + R4) + (R1 + R2 + R3 + R4) R5) ℰ) / (R1 R2 R3 + R1 R2 R4 + R1 R3 R4 + R2 R3 R4 + (R1 + R2) (R3 + R4) R5)}}
```

Since I_3 is the current going through the battery,

$$I_3 = \frac{\mathcal{E}}{R_{eq}} \tag{24}$$

where R_{eq} is the equivalent resistance that we are looking for. Comparing the form of R_{eff} thus found by *Mathematica*, we do indeed see that it is identical to Equation (18), as desired. □

Capacitors

Theory

The power delivered to a circuit element at potential difference V with current I flowing through it equals

$$P = I\, V = I^2\, R \tag{25}$$

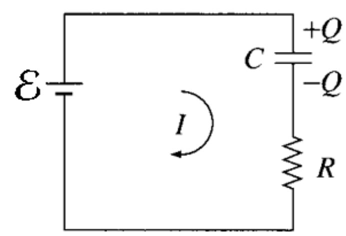
A capacitor has the circuit symbol  and functions by accumulating charge Q until the potential $V = \frac{Q}{C}$ across the capacitor prevents the further flow of current across the capacitor.

One important point to remember is that current never flows between the plates of a capacitor. Rather, current flows from one plate around the circuit into the other plate, leaving the former negatively charged and the latter positively charged. A typical circuit problem will analyze the charging and discharging of a capacitor, and in both cases the current around the circuit eventually slows down to zero.

Charging a Capacitor

<u>Example</u>

You can charge up a capacitor by connecting it to a battery of fixed voltage $\mathcal{E}$. Imagine that we hook up an uncharged capacitor to the circuit below at time $t = 0$.



1. Determine $Q[t]$ and $I[t]$ as functions of time.
2. Find the total energy output of the battery ($\int I[t]\,\mathcal{E}\,dt$). Determine the heat delivered to the resistor.
3. What is the final energy stored in the capacitor? What fraction of the work done by the battery shows up as energy in the capacitor? (Notice that the answer is independent of R!)

Solution

1. Let $Q[t]$ be the charge on the capacitor, so that the current $I = \frac{dQ[t]}{dt}$. The loop equation around the circuit equals

$$\begin{aligned} 0 &= \mathcal{E} - \frac{Q}{C} - I\,R \\ &= \mathcal{E} - \frac{Q}{C} - \frac{dQ}{dt}\,R \end{aligned} \tag{26}$$

Solving this differential equation yields

$$Q[t] = C\,\mathcal{E} + c_1\,e^{-\frac{t}{R\,C}} \tag{27}$$

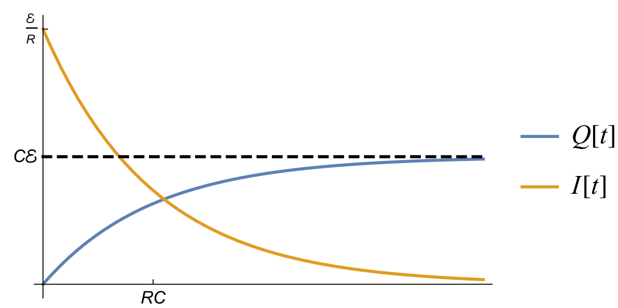
Upon substituting the initial condition $Q[0] = 0$, $c_1 = -C\,\mathcal{E}$ so that

$$Q[t] = C\,\mathcal{E}\left(1 - e^{-\frac{t}{R\,C}}\right) \tag{28}$$

Therefore, the current in the circuit equals

$$I[t] = \frac{dQ[t]}{dt} = \frac{\mathcal{E}}{R}\,e^{-\frac{t}{R\,C}} \tag{29}$$

The plot below shows $Q[t]$ and $I[t]$ for the case when $\frac{\mathcal{E}}{R} > C\,\mathcal{E}$. Notice that the capacitor will charge up until its potential asymptotically reaches $\mathcal{E}$, at which point current will stop flowing.



2. The total energy delivered from the battery equals

$$\begin{aligned} W_{\text{battery}} &= \int_0^{\infty} I[t]\,\mathcal{E}\,dt \\ &= \int_0^{\infty} \frac{dQ[t]}{dt}\,\mathcal{E}\,dt \\ &= \mathcal{E}\,(Q[\infty] - Q[0]) \\ &= C\,\mathcal{E}^2 \end{aligned} \tag{30}$$

The heat delivered to the resistor equals

$$\begin{aligned} W_{\text{resistor}} &= \int_0^\infty I[t]^2 R\, dt \\ &= \int_0^\infty \frac{\mathcal{E}^2}{R} e^{-\frac{2t}{RC}} dt \\ &= \frac{1}{2} C \mathcal{E}^2 \end{aligned} \tag{31}$$

3. The energy in a capacitor equals $\frac{1}{2}\frac{Q^2}{C}$. Thus, the capacitor started out with zero energy and through the charging process gained the energy

$$\begin{aligned} W_{\text{capacitor}} &= \frac{1}{2}\frac{Q[\infty]^2}{C} \\ &= \frac{1}{2} C \mathcal{E}^2 \end{aligned} \tag{32}$$

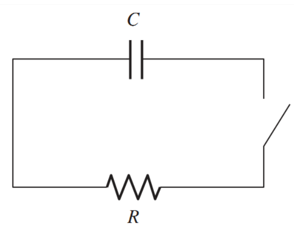
Therefore, half of the battery's work went into charging the capacitor and half was dissipated by the resistor (there is no free lunch!). It is remarkable that this result is independent of the resistance R! □

Advanced Section: A Superconducting Circuit

Discharging a Capacitor

<u>Example</u>

The book (Section 4.11) describes a the simple RC circuit shown below. Assuming that the capacitor starts off with charge Q_0 and the switch is closed at $t = 0$,



$$Q[t] = C V_0 e^{-\frac{t}{RC}} \tag{33}$$

$$I[t] = -\frac{dQ[t]}{dt} = \frac{V_0}{R} e^{-\frac{t}{RC}} \tag{34}$$

Find the original energy stored in the capacitor. Then integrate the total heat delivered to the resistor over time and verify that this equals the energy lost by the capacitor at any time t.

<u>Solution</u>

Note that the current $I[t] = -\frac{dQ[t]}{dt}$ which is the *negative* of the relation found in the "Charging a Capacitor" problem above because the capacitor is discharging!

The energy stored in a capacitor equals

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2} C V^2 \tag{35}$$

Assume (without loss of generality) that the right plate of the capacitor is positively charged while the left plate is negatively charged, so that current will flow clockwise. The loop equation (written in terms of Q) equals

$$0 = \frac{Q}{C} - I R = \frac{Q}{C} + \frac{dQ}{dt} R \tag{36}$$

which has the solution

$$Q[t] = c_0 e^{-\frac{t}{RC}} \tag{37}$$

Using the initial condition $Q[0] = Q_0$, $c_0 = Q_0$ so that

$$Q[t] = Q_0 e^{-\frac{t}{RC}} \tag{38}$$

At any time t, the power delivered to the resistor equals

$$\begin{aligned} P &= I^2 R \\ &= \left(\frac{dQ}{dt}\right)^2 R \\ &= \frac{Q_0^2}{R C^2} e^{-\frac{2t}{RC}} \end{aligned} \tag{39}$$

Therefore, the total energy dissipated by the resistor equals

$$\int_0^t P\, dt = \int_0^t \frac{Q_0^2}{R C^2} e^{-\frac{2t}{RC}}\, dt = \frac{Q_0^2}{2C}\left(1 - e^{-\frac{2t}{RC}}\right) \tag{40}$$

while the energy lost by the capacitor equals

$$\begin{aligned} \Delta U &= \frac{1}{2}\frac{Q_0^2}{C} - \frac{1}{2}\frac{Q[t]^2}{C} \\ &= \frac{1}{2}\frac{Q_0^2}{C} - \frac{1}{2}\frac{Q_0^2 e^{-\frac{2t}{RC}}}{C} \\ &= \frac{Q_0^2}{2C}\left(1 - e^{-\frac{2t}{RC}}\right) \end{aligned} \tag{41}$$

in agreement with Equation (40), as expected. □